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Generalized (η, ρ)-Invex Functions and Global Semiparametric Sufficient Efficiency Conditions for Multiobjective Fractional Programming Problems Containing Arbitrary Norms

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Abstract. The purpose of this paper is to develop a fairly large number of sets of global semiparametric sufficient efficiency conditions under various generalized (η, ρ)-invexity assumptions for a multiobjective fractional programming problem involving arbitrary norms.

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1. Introduction

In this paper, we present a multitude of global semiparametric sufficient efficiency results under a variety of generalized (η, ρ)-invexity conditions for the following multiobjective fractional programming problem involving nondifferentiable functions:

$$(P) \qquad \text{Minimize} \left(\frac{f_1(x) + \|A_1 x\|_{a(1)}}{g_1(x) - \|B_1 x\|_{b(1)}}, \ldots, \frac{f_p(x) + \|A_p x\|_{a(p)}}{g_p(x) - \|B_p x\|_{b(p)}} \right)$$

subject to

$$G_j(x) + \|C_j x\|_{c(j)} \leqq 0, \quad j \in \underline{q}, \qquad H_k(x) = 0, \quad k \in \underline{r}, \quad x \in X,$$

where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i, g_i, i \in \underline{p} \equiv \{1, 2, \ldots, p\}, G_j, j \in \underline{q}$, and $H_k, k \in \underline{r}$, are real-valued functions defined on X, for each $i \in \underline{p}$ and each $j \in \underline{q}$, A_i, B_i, and C_j are, respectively, $\ell_i \times n$, $m_i \times n$, and $n_j \times n$ matrices, $\| \cdot \|_{a(i)}$, $\| \cdot \|_{b(i)}$, and $\| \cdot \|_{c(j)}$ are arbitrary norms in $\mathbb{R}^{\ell_i}$, $\mathbb{R}^{m_i}$, and $\mathbb{R}^{n_j}$, respectively, and for each $i \in \underline{p}$, $g_i(x) - \|B_i x\|_{b(i)} > 0$ for all x satisfying the constraints of (P).

Several classes of static and dynamic optimization problems with multiple fractional objective functions have been the subject of intense investigations in the past few years, which have produced a number of sufficiency and duality results for these problems. Fairly extensive lists of references pertaining to various aspects of multiobjective fractional programming are available in [22–25]. For more information about the vast general area of multiobjective programming, the reader may consult [11, 17, 18, 20].

A close examination of these and other related sources will readily reveal the fact that so far multiobjective fractional programming problems containing arbitrary norms in their objective functions and constraints have not been studied in the area of multiobjective programming. It is our intention to fully investigate the efficiency and duality aspects of these important and interesting nonlinear programming models in a series of papers. We shall continue our investigation here by establishing a fairly large number of global semiparametric sufficient efficiency results for (P). The relevance and applicability of these results to the construction and analysis of several major semiparametric duality models for (P) are demonstrated in [Zalmai, submitted], and their parametric counterparts are presented in [Zalmai, submitted].

The rest of this paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we begin our discussion of sufficient efficiency conditions where we formulate and prove numerous sets of efficiency criteria under a variety of generalized (η, ρ)-invexity assumptions that are placed on the individual as well as certain combinations of the problem functions. Utilizing two partitioning schemes, in Section 4 we establish several sets of generalized sufficiency results each of which is in fact a family of such results whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 5 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this paper.

It is evident that all the sufficient efficiency results obtained for (P) are also applicable, when appropriately specialized, to the following ten classes of problems with multiple, fractional, and conventional objective functions, which are particular cases of (P):

$$(P1) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \left(f_1(x) + \|A_1x\|_{a(1)}, \ldots, f_p(x) + \|A_px\|_{a(p)}\right);$$

$$(P2) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \frac{f_1(x) + \|A_1x\|_{a(1)}}{g_1(x) - \|B_1x\|_{b(1)}};$$

$$(P3) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}}\; f_1(x) + \|A_1 x\|_{a(1)},$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x) + \|C_j x\|_{c(j)} \leqq 0,\ j \in \underline{q},\ \ H_k(x) = 0,\ k \in \underline{r}\};$$

$$(P4) \qquad \text{Minimize}\; \left(\frac{f_1(x) + \langle x, P_1 x\rangle^{1/2}}{g_1(x) - \langle x, Q_1 x\rangle^{1/2}}, \dots, \frac{f_p(x) + \langle x, P_p x\rangle^{1/2}}{g_p(x) - \langle x, Q_p x\rangle^{1/2}}\right)$$

subject to

$$G_j(x) + \langle x, R_j x\rangle^{1/2} \leqq 0,\quad j \in \underline{q}, \qquad H_k(x) = 0,\ \ k \in \underline{r},\ \ x \in X,$$

where P_i, Q_i, $i \in \underline{p}$, and R_j, $j \in \underline{q}$, are $n \times n$ symmetric positive semidefinite matrices, $\langle u, v\rangle$ denotes the inner (scalar) product of the ν-dimensional vectors u and v, that is, $\langle u, v\rangle = \sum_{i=1}^{\nu} u_i v_i$, where u_i and v_i are the ith components of u and v, respectively, and for each $i \in \underline{p}$, $g_i(x) - \langle x, Q_i x\rangle^{1/2} > 0$ for all feasible solutions of (P4);

$$(P5) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}}\; \left(f_1(x) + \langle x, P_1 x\rangle^{1/2}, \dots, f_p(x) + \langle x, P_p x\rangle^{1/2}\right);$$

$$(P6) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}}\; \frac{f_1(x) + \langle x, P_1 x\rangle^{1/2}}{g_1(x) - \langle x, Q_1 x\rangle^{1/2}};$$

$$(P7) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}}\; f_1(x) + \langle x, P_1 x\rangle^{1/2},$$

where $\mathbb{G}$ is the feasible set of (P4), that is,

$$\mathbb{G} = \{x \in X : G_j(x) + \langle x, R_j x\rangle^{1/2} \leqq 0,\ j \in \underline{q}, \quad H_k(x) = 0,\ k \in \underline{r}\};$$

$$(P8) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}}\; \left(f_1(x), \dots, f_p(x)\right);$$

$$(P9) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}}\; \frac{f_1(x)}{g_1(x)};$$

$$(P10) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}}\; f_1(x),$$

where $\mathbb{H} = \{x \in X : G_j(x) \leqq 0,\ j \in \underline{q},\ H_k(x) = 0,\ k \in \underline{r}\}$.

The problems $(P4), (P5), (P6)$, and $(P7)$ are special cases of (P), $(P1), (P2)$, and $(P3)$, respectively, which are obtained by choosing $\|\cdot\|_{a(i)}$, $\|\cdot\|_{b(i)}, i \in \underline{p}$, and $\|\cdot\|_{c(j)}, j \in \underline{q}$, to be the ℓ_2-norm $\|\cdot\|_2$, and defining $P_i = A_i^T A_i$, $Q_i = B_i^T B_i, i \in \underline{p}$, and $R_j = C_j^T C_j, j \in \underline{q}$.

Since in most cases these results can easily be modified and restated for each one of the above ten problems, we shall not state them explicitly.

Optimization problems containing norms arise naturally in many areas of the decision sciences, applied mathematics, and engineering. They are encountered most frequently in facility location problems, approximation theory, and engineering design. A number of these problems have already been investigated in the related literature. Similarly, optimization problems involving square roots of positive semidefinite quadratic forms have arisen in stochastic programming, multifacility location problems, and portfolio selection problems, among others. A fairly extensive list of references pertaining to several aspects of these two classes of problems is given in [21].

2. Preliminaries

In this section we recall, for convenience of reference, the definitions of certain classes of generalized convex functions which will be needed in the sequel. We begin by defining an invex function which has been instrumental in creating a vast array of interesting and important classes of generalized convex functions.

DEFINITION 2.1. Let f be a real-valued differentiable function defined on X. Then f is said to be η-invex at y if there exists a function $\eta: X \times X \to \mathbb{R}^n$ such that for each $x \in X$,

$$f(x) - f(y) \geqq \langle \nabla f(y), \eta(x, y) \rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)^T$ is the gradient of f at y and the superscript T signifies transposition; f is said to be η-invex on X if the above inequality holds for all $x, y \in X$.

From this definition it is clear that every real-valued differentiable convex function is invex with $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [5] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leqq 0,\ i \in \underline{m},\ x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i: \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the same function $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush–Kuhn–Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [2] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need a simple extension of invexity, namely, ρ-invexity which was originally defined in [8].

Let η be a function from $X \times X$ to $\mathbb{R}^n$, and let h be a real-valued differentiable function defined on X.

DEFINITION 2.2. The function h is said to be (strictly) (η, ρ)-invex at x^* if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x) - h(x^*)(>) \geqq \langle \nabla h(x^*), \eta(x, x^*) \rangle + \rho \|x - x^*\|^2.$$

DEFINITION 2.3. The function h is said to be (prestrictly) (η, ρ)-*quasiinvex* at $x^* \in X$ if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x)(<) \leqq h(x^*) \Rightarrow \langle \nabla h(x^*), \eta(x, x^*) \rangle \leqq -\rho \|x - x^*\|^2.$$

DEFINITION 2.4. The function h is said to be (strictly) (η, ρ)-*pseudoinvex* at $x^* \in X$ if there exists $\rho \in \mathbb{R}$ such that for each $x \in X (x \neq x^*)$,

$$\langle \nabla h(x^*), \eta(x, x^*) \rangle \geqslant -\rho \|x - x^*\|^2 \Rightarrow h(x)(>) \geqslant h(x^*).$$

From the above definitions it is clear that if h is (η, ρ)-invex at x^*, then it is both (η, ρ)-quasiinvex and (η, ρ)-pseudoinvex at x^*, if h is (η, ρ)-quasiinvex at x^*, then it is prestrictly (η, ρ)-quasiinvex at x^*, and if h is strictly (η, ρ)-pseudoinvex at x^*, then it is (η, ρ)-quasiinvex at x^*.

In the proofs of the sufficiency theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, (η, ρ)-pseudoinvexity can be defined in the following equivalent way: The function h is said to be (η, ρ)-pseudoinvex at x^* if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x) < h(x^*) \Rightarrow \langle \nabla h(x^*), \eta(x, x^*) \rangle < -\rho \|x - x^*\|^2.$$

The concept of ρ-invexity has been extended in many ways, and various types of generalized ρ-invex functions have been utilized for establishing a variety of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult [1–4, 6, 10, 12, 14, 16], and for recent surveys of these and related functions, the reader is referred to [9, 15].

In the remainder of this section, we present a set of semiparametric necessary efficiency conditions for (P). We begin by introducing a consistent notation for vector inequalities. For $a, b \in \mathbb{R}^m$, the following order notation will be used: $a \geqq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$; $a \geqslant b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$, but $a \neq b$; $a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$; and $a \ngeqslant b$ is the negation of $a \geqslant b$.

Consider the multiobjective problem

$$(P^*) \qquad \underset{x \in \mathcal{X}}{\text{Minimize}}\, F(x) = (F_1(x), \ldots, F_p(x)),$$

where F_i, $i \in \underline{p}$, are real-valued functions defined on the set $\mathcal{X}$.

An element $x^\circ \in \mathcal{X}$ is said to be an *efficient (Pareto optimal, nondominated, noninferior)* solution of (P^*) if there exists no $x \in \mathcal{X}$ such that $F(x) \leqslant F(x^\circ)$.

Throughout the sequel, we shall assume that the problem functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are continuously differentiable on the open set X.

THEOREM 2.1 [Zalmai, Submitted]. *Let x^* be a normal efficient solution of (P) (i.e., an efficient solution of (P) at which a suitable constraint qualification holds) and let $\lambda_i^* = \varphi_i(x^*)$, $i \in \underline{p}$. Then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p, u > 0, \sum_{i=1}^p u_i = 1\}$, $v^* \in \mathbb{R}_+^q \equiv \{v \in \mathbb{R}^q, v \geqq 0\}$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^p u_i^* \{\nabla f_i(x^*) + A_i^T \alpha^{*i} - \lambda_i^* [\nabla g_i(x^*) - B_i^T \beta^{*i}]\} + \sum_{j=1}^q v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^r w_k^* \nabla H_k(x^*) = 0,$$

$$v_j^* [G_j(x^*) + \|C_j x^*\|_{c(j)}] = 0, \quad j \in \underline{q},$$

$$\|\alpha^{*i}\|_{a(i)}^* \leqq 1, \quad \|\beta^{*i}\|_{b(i)}^* \leqq 1, \quad i \in \underline{p},$$

$$\|\gamma^{*j}\|^*_{c(j)} \leqq 1, \quad j \in \underline{q},$$

$$\langle \alpha^{*i}, A_i x^* \rangle = \|A_i x^*\|_{a(i)}, \quad \langle \beta^{*i}, B_i x^* \rangle = \|B_i x^*\|_{b(i)}, \quad i \in \underline{p},$$

$$\langle \gamma^{*j}, C_j x^* \rangle = \|C_j x^*\|_{c(j)}, \quad j \in \underline{q},$$

where for each $i \in \underline{p}$, φ_i *is the* ith *objective function of (P), and* $\|\cdot\|^*_a$ *is the dual of the norm* $\|\cdot\|_a$, *that is,* $\|\delta\|^*_a = \max_{\|\xi\|_a=1} |\langle \delta, \xi \rangle|$.

The necessary efficiency conditions in Theorem 2.1 contain two sets of parameters, namely, $\{u^*_i\}$ and $\{\lambda^*_i\}$, $i \in \underline{p}$, which were introduced as a consequence of our indirect approach utilized in [Zalmai, submitted] requiring two auxiliary intermediate problems. It is possible to eliminate one of these two sets of parameters and thus obtain a *semiparametric* version of Theorem 2.1. Indeed, substituting $N_i(x^*)/D_i(x^*)$, for $\lambda^*_i = \varphi_i(x^*)$, where $N_i(x^*) = f_i(x^*) + \|A_i x^*\|_{a(i)}$ and $D_i(x^*) = g_i(x^*) - \|B_i x^*\|_{b(i)}$, $i \in \underline{p}$, simplifying, and redefining the multiplier vectors, we obtain the following semiparametric form of Theorem 2.1.

THEOREM 2.2. *Let* x^* *be a normal efficient solution of (P). Then there exist* $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, *and* $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, *such that*

$$\sum_{i=1}^{p} u^*_i \{D_i(x^*)[\nabla f_i(x^*) + A^T_i \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B^T_i \beta^{*i}]\} + \sum_{j=1}^{q} v^*_j [\nabla G_j(x^*) + C^T_j \gamma^{*j}] + \sum_{k=1}^{r} w^*_k \nabla H_k(x^*) = 0, \tag{2.1}$$

$$v^*_j [G_j(x^*) + \|C_j x^*\|_{c(j)}] = 0, \quad j \in \underline{q}, \tag{2.2}$$

$$\|\alpha^{*i}\|^*_{a(i)} \leqq 1, \quad \|\beta^{*i}\|^*_{b(i)} \leqq 1, \quad i \in \underline{p}, \tag{2.3}$$

$$\|\gamma^{*j}\|^*_{c(j)} \leqq 1, \quad j \in \underline{q}, \tag{2.4}$$

$$\langle \alpha^{*i}, A_i x^* \rangle = \|A_i x^*\|_{a(i)}, \quad \langle \beta^{*i}, B_i x^* \rangle = \|B_i x^*\|_{b(i)}, \quad i \in \underline{p}, \tag{2.5}$$

$$\langle \gamma^{*j}, C_j x^* \rangle = \|C_j x^*\|_{c(j)}, \quad j \in \underline{q}. \tag{2.6}$$

We conclude this section by specializing Theorem 2.2 for problem $(P4)$. This will provide a glimpse of what is involved in modifying and restating the results of this paper for $(P4)$–$(P7)$.

THEOREM 2.3. *Let x^* be a normal efficient solution of (P). Then there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\xi^{*i}, \pi^{*i}, \sigma^{*j} \in \mathbb{R}^n, i \in \underline{p}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p} u_i^* \{\nu_i(x^*)[\nabla f_i(x^*) + K_i \xi^{*i}] - \delta_i(x^*)[\nabla g_i(x^*) - L_i \pi^{*i}]\} + \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + M_j \sigma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0,$$

$$v_j^*[G_j(x^*) + \langle x^*, M_j x^* \rangle^{1/2}] = 0, \quad j \in \underline{q},$$

$$\langle \xi^{*i}, K_i \xi^{*i} \rangle \leqq 1, \quad \langle \pi^{*i}, L_i \pi^{*i} \rangle \leqq 1, \quad i \in \underline{p},$$

$$\langle \sigma^{*j}, M_j \sigma^{*j} \rangle \leqq 1, \quad j \in \underline{q},$$

$$\langle \xi^{*i}, K_i x^{*i} \rangle = \langle x^{*i}, K_i x^{*i} \rangle^{1/2}, \quad \langle \pi^{*i}, L_i x^{*i} \rangle = \langle x^{*i}, L_i x^{*i} \rangle^{1/2}, \quad i \in \underline{p},$$

$$\langle \sigma^{*j}, M_j x^* \rangle = \langle x^*, M_j x^* \rangle^{1/2}, \quad j \in \underline{q},$$

where $\nu_i(x^*) = f_i(x) + \langle x, P_i x \rangle^{1/2}$ and $\delta_i(x^*) = g_i(x) - \langle x, Q_i x \rangle^{1/2}$, $i \in \underline{p}$.

The form and contents of the necessary efficiency conditions given in Theorem 2.2 provide clear guidelines for formulating numerous sets of sufficient efficiency conditions and many duality models for (P). The rest of this paper is devoted to investigating various sets of sufficiency criteria for (P). A vast number of duality results for (P) which are based on these sufficiency results and Theorem 2.2 are discussed in [Zalmai, Submitted].

3. Sufficient efficiency conditions

In this section, we present a fairly large number of semiparametric sufficient efficiency results in which various generalized (η, ρ)-invexity assumptions are imposed on the individual as well as certain combinations of the problem functions. In formulating these results, we shall use a streamlined version of the necessary efficiency conditions given in Theorem 2.2. Specifically, we use a slightly altered version of these conditions obtained by dropping (2.5) and modifying (2.2) accordingly. The resulting reduced set of equations and inequalities will lead to relatively shorter statements and proofs for many of the sufficiency principles that will be developed in this study.

In the proofs of our sufficiency theorems, we shall make frequent use of the well-known generalized Cauchy inequality which is formally stated in the following lemma.

LEMMA 3.1 [7]. *For each* $a, b \in \mathbb{R}^m$, $\langle a, b \rangle \leqq \|a\|^* \|b\|$.

To simplify the ensuing presentation, we use the following list of symbols:

$$\mathcal{A}_i(x, \alpha) = f_i(x) + \langle \alpha^i, A_i x \rangle, \quad i \in \underline{p},$$
$$\mathcal{B}_i(x, \beta) = -g_i(x) + \langle \beta^i, B_i x \rangle, \quad i \in \underline{p},$$
$$\mathcal{C}_j(x, \gamma) = G_j(x) + \langle \gamma^j, C_j x \rangle, \quad j \in \underline{q},$$
$$\mathcal{D}_k(x, w) = w_k H_k(x),\ k \in \underline{r},$$
$$\mathcal{E}_i(x, x^*, \alpha, \beta) = D_i(x^*)[f_i(x) + \langle \alpha^i, A_i x \rangle] - N_i(x^*)[g_i(x) - \langle \beta^i, B_i x \rangle], \quad i \in \underline{p},$$
$$\mathcal{C}(x, v, \gamma) = \sum_{j=1}^{q} v_j [G_j(x) + \langle \gamma^j, C_j x \rangle],$$
$$\mathcal{D}(x, w) = \sum_{k=1}^{r} w_k H_k(x),$$
$$\mathcal{E}(x, x^*, u, \alpha, \beta) = \sum_{i=1}^{p} u_i \{D_i(x^*)[f_i(x) + \langle \alpha^i, A_i x \rangle] - N_i(x^*)[g_i(x) - \langle \beta^i, B_i x \rangle]\},$$
$$\mathcal{F}(x, v, w, \gamma) = \sum_{j=1}^{q} v_j [G_j(x) + \langle \gamma^j, C_j x \rangle] + \sum_{k=1}^{r} w_k H_k(x),$$
$$I_+(u) = \{i \in \underline{p} : u_i > 0\} \text{ for fixed } u \in U,$$
$$J_+(v) = \{j \in \underline{q} : v_j > 0\} \text{ for fixed } v \in \mathbb{R}^q_+,$$

$$K_*(w) = \{k \in \underline{r} : w_k \neq 0\} \text{ for fixed } w \in \mathbb{R}^r,$$
$$U_0 = \{u \in \mathbb{R}^p_+ : \sum_{i=1}^{p} u_i = 1\},$$
$$\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^p),$$
$$\beta = (\beta^1, \beta^2, \ldots, \beta^p),$$
$$\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^q).$$

We begin our discussion of sufficiency criteria for (P) with a collection of results in which separate (η, ρ)-invexity conditions are imposed on the functions $\mathcal{A}_i(\cdot, \alpha)$ and $\mathcal{B}_i(\cdot, \beta)$, $i \in \underline{p}$, whereas different types of generalized (η, ρ)-invexity assumptions are placed on certain combinations of the constraint functions.

THEOREM 3.1. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, and that there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\} + \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \tag{3.1}$$

$$v_j^*[G_j(x^*) + \langle \gamma^{*j}, C_j x^* \rangle] = 0, \quad j \in \underline{q}, \tag{3.2}$$

$$\|\alpha^{*i}\|_{a(i)}^* \leqq 1, \quad \|\beta^{*i}\|_{b(i)}^* \leqq 1, \quad i \in \underline{p}, \tag{3.3}$$

$$\|\gamma^{*j}\|_{c(j)}^* \leqq 1, \quad j \in \underline{q}, \tag{3.4}$$

$$\langle \alpha^{*i} A_i x^* \rangle = \|A_i x^*\|_{a(i)}, \quad \langle \beta^{*i} B_i x^* \rangle = \|B_i x^*\|_{b(i)}, \quad i \in \underline{p}. \tag{3.5}$$

Assume, furthermore, that any one of the following six sets of conditions holds:

(a) *(i) for each* $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha^*)$ *is* $(\eta, \bar{\rho}_i)$*-invex and* $\mathcal{B}_i(\cdot, \beta^*)$ *is* $(\eta, \tilde{\rho}_i)$*-invex at* x^**;*

(ii) for each $j \in J_+ \equiv J_+(v^*)$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*

(iii) for each $k \in K_* \equiv K_*(w^*)$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^**;*

(iv) $\rho^* + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*, where* $\rho^* = \sum_{i=1}^{p} u_i^*[D_i(x^*)\bar{\rho}_i + N_i(x^*)\tilde{\rho}_i]$*;*

(b) *(i) for each* $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha^*)$ *is* $(\eta, \bar{\rho}_i)$*-invex and* $\mathcal{B}_i(\cdot, \beta^*)$ *is* $(\eta, \tilde{\rho}_i)$*-invex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^**;*

(iii) for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^**;*

(iv) $\rho^* + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*;*

(c) *(i) for each* $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha^*)$ *is* $(\eta, \bar{\rho}_i)$*-invex and* $\mathcal{B}_i(\cdot, \beta^*)$ *is* $(\eta, \tilde{\rho}_i)$*-invex at* x^**;*

(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*

(iv) $\rho^* + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \breve{\rho} \geqq 0$*;*

(d) *(i) for each* $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha^*)$ *is* $(\eta, \bar{\rho}_i)$*-invex and* $\mathcal{B}_i(\cdot, \beta^*)$ *is* $(\eta, \tilde{\rho}_i)$*-invex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*

(iv) $\rho^* + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(e) *(i) for each* $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \alpha^*)$ *is* $(\eta, \bar{\rho}_i)$*-invex and* $\mathcal{B}_i(\cdot, \beta^*)$ *is* $(\eta, \tilde{\rho}_i)$*-invex at* x^**;*

(ii) $\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^**;*

(iii) $\rho^* + \hat{\rho} \geqq 0$*;*

(f) the Lagrangian-type function

$$z \to L(z, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) = \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(z) + \langle \alpha^{*i}, A_i z \rangle]$$

$$-N_i(x^*)[g_i(z) - \langle \beta^{*i}, B_i z \rangle]\} + \sum_{j=1}^{q} v_j^*[G_j(z) + \langle \gamma^{*j}, C_j z \rangle] + \sum_{k=1}^{r} w_k^* H_k(z)$$

is $(\eta, 0)$*-pseudoinvex at* x^**.*

Then x^* *is an efficient solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).

(a): Since for each $j \in J_+$,

$$\begin{aligned} G_j(x)+\langle\gamma^{*j}, C_jx\rangle &\leqq G_j(x)+\|\gamma^{*j}\|^*_{c(j)}\|C_jx\|_{c(j)} \quad \text{(by Lemma 3.1)} \\ &\leqq G_j(x)+\|C_jx\|_{c(j)} \quad \text{(by (3.4))} \\ &\leqq 0 \ \text{(since } x\in\mathbb{F}) \\ &= G_j(x^*)+\langle\gamma^{*j}, C_jx^*\rangle \quad \text{(by (3.2))}, \end{aligned}$$

in view of (ii) we have

$$\langle\nabla G_j(x^*)+C_j^T\gamma^{*j}, \eta(x,x^*)\rangle \leqq -\hat{\rho}_j\|x-x^*\|^2.$$

As $v_j^* \geqq 0$ for each $j\in\underline{q}$, and $v_j^*=0$ for each $j\in\underline{q}\backslash J_+$ (complement of J_+ relative to $\underline{q}$), the above inequalities yield

$$\left\langle\sum_{j=1}^{q} v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}], \eta(x,x^*)\right\rangle \leqq -\sum_{j\in J_+} v_j^*\hat{\rho}_j\|x-x^*\|^2. \tag{3.6}$$

In a similar manner we can show that (iii) leads to the following inequality:

$$\left\langle\sum_{k=1}^{r} w_k^*\nabla H_k(x^*), \eta(x,x^*)\right\rangle \leqq -\sum_{k\in K_*}\breve{\rho}_k\|x-x^*\|^2. \tag{3.7}$$

Keeping in mind that $u^*>0$, $N_i(x^*)\geqq 0$, and $D_i(x^*)>0$, we have

$$\begin{aligned} &\sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\|A_ix\|_{a(i)}]-N_i(x^*)[g_i(x)-\|B_ix\|_{b(i)}]\} \\ &\geqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\|\alpha^{*i}\|^*_{a(i)}\|A_ix\|_{a(i)}] \\ &\qquad -N_i(x^*)[g_i(x)-\|\beta^{*i}\|^*_{b(i)}\|B_ix\|_{b(i)}]\} \quad \text{(by (3.3))} \\ &\geqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\langle\alpha^{*i}, A_ix\rangle] \\ &\qquad -N_i(x^*)[g_i(x)-\langle\beta^{*i}, B_ix\rangle]\} \quad \text{(by Lemma 3.1)} \\ &= \sum_{i=1}^{p} u_i^*\{D_i(x^*)\{f_i(x)+\langle\alpha^{*i}, A_ix\rangle-[f_i(x^*)+\langle\alpha^{*i}, A_ix^*\rangle]\} \\ &\qquad -N_i(x^*)\{g_i(x)-\langle\beta^{*i}, B_ix\rangle-[g_i(x^*)-\langle\beta^{*i}, B_ix^*\rangle]\}\} \\ &\quad \text{(by the definitions of } N_i(x^*) \text{ and } D_i(x^*),\ i\in\underline{p}, \text{ and (3.5))} \end{aligned}$$

$$
\begin{aligned}
&\geqq \sum_{i=1}^{p} u_i^* \{\langle D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}], \eta(x, x^*)\rangle \\
&\qquad + [D_i(x^*)\bar{\rho}_i + N_i(x^*)\tilde{\rho}_i]\|x - x^*\|^2\} \quad \text{(by (i))} \\
&= -\Big\langle \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] \\
&\qquad + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*), \eta(x, x^*)\Big\rangle + \sum_{i=1}^{p} u_i^* [D_i(x^*)\bar{\rho}_i \\
&\quad + N_i(x^*)\tilde{\rho}_i]\|x - x^*\|^2 \quad \text{(by (3.1))} \\
&\geqq \Big(\rho^* + \sum_{j\in J_+} v_j^* \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k\Big)\|x - x^*\|^2 \quad \text{(by (3.6) and (3.7))} \\
&\geqq 0 \quad \text{(by (iv))}.
\end{aligned} \tag{3.8}
$$

Since $u^* > 0$ the above inequality implies that

$$
\big(D_1(x^*)[f_1(x) + \|A_1 x\|_{a(1)}] - N_1(x^*)[g_1(x) - \|B_1 x\|_{b(1)}], \ldots, \\
D_p(x^*)[f_p(x) + \|A_p x\|_{a(p)}] - N_p(x^*)[g_p(x) - \|B_p x\|_{b(p)}]\big) \nleq \big(0, \ldots, 0\big),
$$

which, in turn, implies that

$$
\left(\frac{f_1(x) + \|A_1 x\|_{a(1)}}{g_1(x) - \|B_1 x\|_{b(1)}}, \ldots, \frac{f_p(x) + \|A_p x\|_{a(p)}}{g_p(x) - \|B_p x\|_{b(p)}}\right) \nleq \left(\frac{N_1(x^*)}{D_1(x^*)}, \ldots, \frac{N_p(x^*)}{D_p(x^*)}\right).
$$

Since $N_i(x^*)/D_i(x^*) = \varphi_i(x^*)$, $i \in \underline{p}$, and x was arbitrary, we conclude that x^* is an efficient solution of (P).

(b): As shown in part (a), for each $j \in J_+$, we have $G_j(x) + \langle \gamma^{*j}, C_j x\rangle \leqq G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle$ and hence

$$
\sum_{j=1}^{q} v_j^* [G_j(x) + \langle \gamma^{*j}, C_j x\rangle] \leqq \sum_{j=1}^{q} v_j^* [G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle],
$$

which in view of (ii) implies that

$$
\Big\langle \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}], \eta(x, x^*)\Big\rangle \leqq -\hat{\rho}\|x - x^*\|^2.
$$

Now proceeding as in the proof of part (a) and using this inequality instead of (3.6), we arrive at (3.8), which leads to the conclusion that x^* is an efficient solution of (P).

(c)–(e): The proofs are similar to those of parts (a) and (b).
(f): By our $(\eta, 0)$-pseudoinvexity assumption, (3.1) implies that

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\langle \alpha^{*i}, A_i x\rangle]-N_i(x^*)[g_i(x)-\langle \beta^{*i}, B_i x\rangle]\} \\
&\quad+\sum_{j=1}^{q} v_j^*[G_j(x)+\langle \gamma^{*j}, C_j x\rangle]+\sum_{k=1}^{r} w_k^* H_k(x) \\
&\geqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x^*)+\langle \alpha^{*i}, A_i x^*\rangle]-N_i(x^*)[g_i(x^*)-\langle \beta^{*i}, B_i x^*\rangle]\} \\
&\quad+\sum_{j=1}^{q} v_j^*[G_j(x^*)+\langle \gamma^{*j}, C_j x^*\rangle]+\sum_{k=1}^{r} w_k^* H_k(x^*).
\end{aligned}$$

Because $x^* \in \mathbb{F}$ and (3.2) and (3.5) hold, the right-hand side of the above inequality is equal to zero, and hence we have

$$\begin{aligned}
0 &\leqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\langle \alpha^{*i}, A_i x\rangle]-N_i(x^*)[g_i(x)-\langle \beta^{*i}, B_i x\rangle]\} \\
&\quad+\sum_{j=1}^{q} v_j^*[G_j(x)+\langle \gamma^{*j}, C_j x\rangle] \\
&\leqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\|\alpha^{*i}\|_{a(i)}^*\|A_i x\|_{a(i)}] \\
&\quad-N_i(x^*)[g_i(x)-\|\beta^{*i}\|_{b(i)}^*\|B_i x\|_{b(i)}]\} \\
&\quad+\sum_{j=1}^{q} v_j^*[G_j(x)+\|\gamma^{*j}\|_{c(j)}^*\|C_j x\|_{c(j)}] \quad \text{(by Lemma 3.1)} \\
&\leqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\|A_i x\|_{a(i)}]-N_i(x^*)[g_i(x)-\|B_i x\|_{b(i)}]\} \\
&\quad+\sum_{j=1}^{q} v_j^*[G_j(x)+\|C_j x\|_{c(j)}] \quad \text{(by (3.3) and (3.4))} \\
&\leqq \sum_{i=1}^{p} u_i^*\{D_i(x^*)[f_i(x)+\|A_i x\|_{a(i)}]-N_i(x^*)[g_i(x)-\|B_i x\|_{b(i)}]\} \\
&\quad \text{(by the feasibility of } x\text{)},
\end{aligned}$$

which is precisely (3.8). As seen in the proof of part (a), this inequality leads to the desired conclusion that x^* is an efficient solution of (P). □

In Theorem 3.1, separate (η, ρ)-invexity assumptions were imposed on the functions $\mathcal{A}_i(\cdot, \alpha^*)$ and $\mathcal{B}_i(\cdot, \beta^*), i \in \underline{p}$. In the remainder of this section, we shall formulate a great variety of sufficient efficiency conditions in which various generalized (η, ρ)-invexity requirements will be placed on certain combinations of these functions.

THEOREM 3.2. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, that there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*\bar{j}} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1) – (3.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:*

(a) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho})$*-pseudoinvex at* x^**;*
(ii) for each $j \in J_+ \equiv J_+(v^*)$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*
(iii) for each $k \in K_* \equiv K_*(w^*)$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^**;*
(iv) $\bar{\rho} + \sum_{j\in J_+} v_j^* \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(b) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho})$*-pseudoinvex at* x^**;*
(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^**;*
(iii) for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^**;*
(iv) $\bar{\rho} + \hat{\rho} + \sum_{k\in K_*} \breve{\rho}_k \geqq 0$*;*

(c) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho})$*-pseudoinvex at* x^**;*
(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*
(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*
(iv) $\bar{\rho} + \sum_{j\in J_+} v_j^* \hat{\rho}_j + \breve{\rho} \geqq 0$*;*

(d) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho})$*-pseudoinvex at* x^**;*
(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^**;*
(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*
(iv) $\bar{\rho} + \hat{\rho} + \breve{\rho} \geqq 0$*;*

(e) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho})$*-pseudoinvex at* x^**;*
(ii) $\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho})$*-quasiinvex at* x^**;*
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then x^* is an efficient solution of (P).

Proof. (a): Let x be an arbitrary feasible solution of (P). Because of our assumptions specified in (ii) and (iii), (3.6) and (3.7) remain valid for the present case. Now combining (3.6) and (3.7) with (3.1) and using (iv), we get

$$\Big\langle \sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}, \eta(x, x^*) \Big\rangle$$

$$\geqq \Big(\sum_{j \in J_+} v_j^* \tilde{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \Big) \|x - x^*\|^2 \geqq -\bar{\rho} \|x - x^*\|^2,$$

which in view of (i) implies that $\mathcal{E}(x, x^*, u^*, \alpha^*, \beta^*) \geqq \mathcal{E}(x^*, x^*, u^*, \alpha^*, \beta^*) = 0$, where the equality follows from the definitions of $D_i(x^*)$ and $N_i(x^*)$, $i \in \underline{p}$, and (3.5). Using this inequality, we see that

$$0 \leqq \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(x) + \langle \alpha^{*i}, A_i x\rangle] - N_i(x^*)[g_i(x) - \langle \beta^{*i}, B_i x\rangle]\}$$
$$\leqq \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(x) + \|\alpha^{*i}\|_{a(i)}^* \|A_i x\|_{a(i)}]$$
$$- N_i(x^*)[g_i(x) - \|\beta^{*i}\|_{b(i)}^* \|B_i x\|_{b(i)}]\} \quad \text{(by Lemma 3.1)}$$
$$\leqq \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(x) + \|A_i x\|_{a(i)}] - N_i(x^*)[g_i(x) - \|B_i x\|_{b(i)}]\} \ \text{(by (3.3))},$$

which is precisely (3.8). As seen in the proof of part (a), this inequality leads to the desired conclusion that x^* is an efficient solution of (P).

(b)–(e): The proofs are similar to that of part (a). □

THEOREM 3.3. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, that there exist $u^* \in U$, $v^* \in \mathbb{R}_+^q$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1)–(3.5) hold. Assume, furthermore, that any one of the following 12 sets of hypotheses is satisfied:*

(a) (i) $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^*;
(ii) *for each* $j \in J_+ \equiv J_+(v^*)$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^*;
(iii) *for each* $k \in K_* \equiv K_*(w^*)$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^*;
(iv) $\bar{\rho} + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k > 0$;

(b) (i) $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^*;
(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^*;
(iii) *for each* $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^*;
(iv) $\bar{\rho} + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k > 0$;

(c) (i) $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^*;
(ii) *for each* $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^*;
(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^*;
(iv) $\bar{\rho} + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \breve{\rho} > 0$;

(d) (i) $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^*;
(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \hat{\rho})$*-quasiinvex at* x^*;

(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*

(iv) $\bar{\rho} + \hat{\rho} + \breve{\rho} > 0$*;*

(e) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) $\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho})$*-quasiinvex at* x^**;*

(iii) $\bar{\rho} + \tilde{\rho} > 0$*;*

(f) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is strictly* $(\eta, \hat{\rho}_j)$*-pseudoinvex at* x^**;*

(iii) for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^**;*

(iv) $\bar{\rho} + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*;*

(g) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is strictly* $(\eta, \hat{\rho})$*-pseudoinvex at* x^**;*

(iii) for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is* $(\eta, \breve{\rho}_k)$*-quasiinvex at* x^**;*

(iv) $\bar{\rho} + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*;*

(h) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*

(iii) for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* x^**;*

(iv) $\bar{\rho} + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*;*

(i) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* x^**;*

(iv) $\bar{\rho} + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \breve{\rho} \geqq 0$*;*

(j) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho})$*-pseudoinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*

(iv) $\bar{\rho} + \tilde{\rho} + \breve{\rho} \geqq 0$*;*

(k) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \tilde{\rho})$*-quasiinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* x^**;*

(iv) $\bar{\rho} + \tilde{\rho} + \breve{\rho} \geqq 0$*;*

(l) *(i)* $\mathcal{E}(\cdot, x^*, u^*, \alpha^*, \beta^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) $\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho})$*-pseudoinvex at* x^**;*

(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then x^* *is an efficient solution of (P).*

Proof. Let x be an arbitrary feasible solution of (P).

(a): Because of our assumptions specified in (ii) and (iii), (3.6) and (3.7) remain valid for the present case. From (3.1), (3.6), (3.7), and (iv) we deduce that

$$\Big\langle \sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}, \eta(x, x^*)\Big\rangle$$
$$\geqq \Big(\sum_{j\in J_+} v_j^* \hat{\rho}_j + \sum_{k\in K_*} \breve{\rho}_k\Big) \|x - x^*\|^2 > -\bar{\rho}\|x - x^*\|^2,$$

which in view of (i) implies that $\mathcal{E}(x, x^*, u^*, \alpha^*, \beta^*) \geqq \mathcal{E}(x^*, x^*, u^*, \alpha^*, \beta^*) = 0$, where the equality follows from the definitions of $D_i(x^*)$ and $N_i(x^*)$, $i \in \underline{p}$, and (3.5). As shown in the proof of Theorem 3.1, this inequality leads to the conclusion that x^* is an efficient solution of (P).

(b)–(e) : The proofs are similar to that of part (a).

(f): As shown in the proof of part (a) of Theorem 3.1, for each $j \in J_+$, we have

$$G_j(x) + \langle \gamma^{*j}, C_j x\rangle \leqq G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle,$$

which by (ii) implies that

$$\langle \nabla G_j(x^*) + C_j^T \gamma^{*j}, \eta(x, x^*)\rangle < -\hat{\rho}_j \|x - x^*\|^2.$$

As $v_j^* \geqq 0$ for each $j \in \underline{q}$, and $v_j^* = 0$ for each $j \in \underline{q} \backslash J_+$, the above inequalities yield

$$\Big\langle \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}], \eta(x, x^*)\Big\rangle < -\sum_{j\in J_+} v_j^* \hat{\rho}_j \|x - x^*\|^2.$$

Now combining this inequality with (3.7) (which is valid for the present case because of (iii)) and (3.1), and using (iv), we obtain

$$\Big\langle \sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}, \eta(x, x^*)\Big\rangle$$
$$> \sum_{j\in J_+} v_j^* \hat{\rho}_j \|x - x^*\|^2 \geqq -\bar{\rho}\|x - x^*\|^2,$$

which in view of (i) implies that $\mathcal{E}(x, x^*, u^*, \alpha^*, \beta^*) \geqq \mathcal{E}(x^*, x^*, u^*, \alpha^*, \beta^*) = 0$. As seen in the proof of Theorem 3.1, this leads to the conclusion that x^* is an efficient solution of (P).

(g)–(l): The proofs are similar to that of part (f). □

THEOREM 3.4. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, and that there exist $u^* \in U_0$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1)–(3.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*
(ii) *for each $j \in J_+ \equiv J_+(v^*)$, $\mathcal{C}_j(\cdot, \gamma^*)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at x^*;*
(iii) *for each $k \in K_* \equiv K_*(w^*)$, $\mathcal{D}_k(\cdot, w^*)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$, where $\rho^\circ = \sum_{i \in J_+} u_i^* \bar{\rho}_i$;*

(b) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*
(ii) *$\mathcal{C}(\cdot, v^*, \gamma^*)$ is $(\eta, \hat{\rho})$-quasiinvex at x^*;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$;*

(c) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at x^*;*
(iii) *$\mathcal{D}(\cdot, w^*)$ is $(\eta, \breve{\rho})$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \breve{\rho} \geqq 0$;*

(d) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*
(ii) *$\mathcal{C}(\cdot, v^*, \gamma^*)$ is $(\eta, \hat{\rho})$-quasiinvex at x^*;*
(iii) *$\mathcal{D}(\cdot, w^*)$ is $(\eta, \breve{\rho})$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \hat{\rho} + \breve{\rho} \geqq 0$;*

(e) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*
(ii) *$\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ is $(\eta, \tilde{\rho})$-quasiinvex at x^*;*
(iii) *$\rho^\circ + \tilde{\rho} \geqq 0$.*

Then x^* is an efficient solution of (P).

Proof. (a): Suppose to the contrary that x^* is not an efficient solution of (P). Then there is a feasible solution $\bar{x}$ of (P) such that $\varphi(\bar{x}) \leqslant \varphi(x^*)$. Since $\varphi(x^*) = N_i(x^*)/D_i(x^*)$ for each $i \in \underline{p}$, the last inequality implies that

$$D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}] \leqq 0 \quad \text{for each } i \in \underline{p}, \tag{3.9}$$

and

$$D_m(x^*)[f_m(\bar{x}) + \|A_m\bar{x}\|_{a(m)}] - N_m(x^*)[g_m(\bar{x}) - \|B_m\bar{x}\|_{b(m)}] < 0 \quad \text{for some } m \in \underline{p}. \tag{3.10}$$

Since for each $i \in \underline{p}$,

$$
\begin{aligned}
&D_i(x^*)[f_i(\bar{x}) + \langle \alpha^{*i}, A_i\bar{x}\rangle] - N_i(x^*)[g_i(\bar{x}) - \langle \beta^{*i}, B_i\bar{x}\rangle] \\
&\quad \leqq D_i(x^*)[f_i(\bar{x}) + \|\alpha^{*i}\|^*_{a(i)}\|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|\beta^{*i}\|^*_{b(i)}\|B_i\bar{x}\|_{b(i)}] \\
&\quad \text{(by Lemma 3.1)} \\
&\quad \leqq D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}] \quad \text{(by (3.3))}
\end{aligned}
$$

it follows from the definitions of $N_i(x^*)$ and $D_i(x^*)$, $i \in \underline{p}$, (3.5), (3.9), and (3.10) that

$$
\begin{aligned}
&D_i(x^*)[f_i(\bar{x}) + \langle \alpha^{*i}, A_i\bar{x}\rangle] - N_i(x^*)[g_i(\bar{x}) - \langle \beta^{*i}, B_i\bar{x}\rangle] \\
&\quad \leqq 0 = D_i(x^*)[f_i(x^*) + \langle \alpha^{*i}, A_i x^*\rangle] - N_i(x^*)[g_i(x^*) - \langle \beta^{*i}, B_i x^*\rangle] \\
&\qquad \text{for each } i \in \underline{p}, \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_m(x^*)[f_m(\bar{x}) + \langle \alpha^{*m}, A_m\bar{x}\rangle] - N_m(x^*)[g_m(\bar{x}) - \langle \beta^{*m}, B_m\bar{x}\rangle] \\
&\quad < 0 = D_m(x^*)[f_m(x^*) + \langle \alpha^{*m}, A_m x^*\rangle] - N_m(x^*)[g_m(x^*) - \langle \beta^{*m}, B_m x^*\rangle] \\
&\qquad \text{for some } m \in \underline{p}. \qquad (3.12)
\end{aligned}
$$

In view of (i), (3.11) and (3.12) imply that for each $i \in I_+$,

$$
\begin{aligned}
&\langle D_i(x^*)[\nabla f_i(x^*) + A_i^T\alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T\beta^{*i}], \eta(\bar{x}, x^*)\rangle \\
&\quad < -\bar{\rho}_i\|\bar{x} - x^*\|^2.
\end{aligned}
$$

Because $u^* \geqq 0$, $u_i^* = 0$ for each $i \in \underline{p}\backslash I_+$, and $\sum_{i\in I_+} u_i^* = 1$, we deduce from the above inequalities that

$$
\begin{aligned}
&\Big\langle \sum_{i=1}^{p} u_i^*\{D_i(x^*)[\nabla f_i(x^*) + A_i^T\alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T\beta^{*i}]\}, \eta(\bar{x}, x^*)\Big\rangle \\
&\quad < -\sum_{i\in I_+} u_i^*\bar{\rho}_i\|\bar{x} - x^*\|^2. \qquad (3.13)
\end{aligned}
$$

As shown in the proof of Theorem 3.1, our assumptions in (ii) and (iii) lead to (3.6) and (3.7), respectively, which when combined with (3.1) and (iv) yield

$$
\begin{aligned}
&\Big\langle \sum_{i=1}^{p} u_i^*\{D_i(x^*)[\nabla f_i(x^*) + A_i^T\alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T\beta^{*i}]\}, \eta(\bar{x}, x^*)\Big\rangle \\
&\quad \geqq \Big(\sum_{j\in J_+} v_j^*\hat{\rho}_j + \sum_{k\in K_*}\breve{\rho}_k\Big)\|\bar{x} - x^*\|^2 \geqq -\sum_{i\in I_+} u_i^*\bar{\rho}_i\|\bar{x} - x^*\|^2,
\end{aligned}
$$

which contradicts (3.13). Therefore, we conclude that x^* is an efficient solution of (P).

(b)–(g): The proofs are similar to that of part (a). □

THEOREM 3.5. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, and that there exist $u^* \in U_0$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1) – (3.5) hold. Assume, furthermore, that any one of the following 12 sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *for each $j \in J_+ \equiv J_+(v^*)$, $\mathcal{C}_j(\cdot, \gamma^*)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at x^*;*
(iii) *for each $k \in K_* \equiv K_*(w^*)$, $\mathcal{D}_k(\cdot, w^*)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k > 0$, where $\rho^\circ = \sum_{i \in I_+} u_i^* \bar{\rho}_i$;*

(b) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *$\mathcal{C}(\cdot, v^*, \gamma^*)$ is $(\eta, \hat{\rho})$-quasiinvex at x^*;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k > 0$;*

(c) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ is $(\eta, \hat{\rho}_j)$-quasiinvex at x^*;*
(iii) *$\mathcal{D}(\cdot, w^*)$ is $(\eta, \breve{\rho})$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \breve{\rho} > 0$;*

(d) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *$\mathcal{C}(\cdot, v^*, \gamma^*)$ is $(\eta, \hat{\rho})$-quasiinvex at x^*;*
(iii) *$\mathcal{D}(\cdot, w^*)$ is $(\eta, \breve{\rho})$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \hat{\rho} + \breve{\rho} > 0$;*

(e) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *$\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ is $(\eta, \tilde{\rho})$-quasiinvex at x^*;*
(iii) *$\rho^\circ + \tilde{\rho} > 0$;*

(f) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ is strictly $(\eta, \hat{\rho})$-pseudoinvex at x^*;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$;*

(g) (i) *for each $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ is $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*
(ii) *$\mathcal{C}(\cdot, v^*, \gamma^*)$ is strictly $(\eta, \hat{\rho})$-pseudoinvex at x^*;*
(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ is $(\eta, \breve{\rho}_k)$-quasiinvex at x^*;*
(iv) *$\rho^\circ + \hat{\rho} + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$;*

(h) *(i) for each* $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* x^**;*

(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*

(iii) for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ *is strictly* $(\eta, \breve{\rho}_k)$*-pseudoinvex at* x^**;*

(iv) $\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \sum_{k \in K_*} \breve{\rho}_k \geqq 0$*;*

(i) *(i) for each* $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* x^**;*

(ii) for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ *is* $(\eta, \hat{\rho}_j)$*-quasiinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* x^**;*

(iv) $\rho^\circ + \sum_{j \in J_+} v_j^* \hat{\rho}_j + \breve{\rho} \geqq 0$*;*

(j) *(i) for each* $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho})$*-pseudoinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is* $(\eta, \breve{\rho})$*-quasiinvex at* x^**;*

(iv) $\rho^\circ + \tilde{\rho} + \breve{\rho} \geqq 0$*;*

(k) *(i) for each* $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* x^**;*

(ii) $\mathcal{C}(\cdot, v^*, \gamma^*)$ *is* $(\eta, \tilde{\rho})$*-quasiinvex at* x^**;*

(iii) $\mathcal{D}(\cdot, w^*)$ *is strictly* $(\eta, \breve{\rho})$*-pseudoinvex at* x^**;*

(iv) $\rho^\circ + \tilde{\rho} + \breve{\rho} \geqq 0$*;*

(l) *(i) for each* $i \in I_+$, $\mathcal{E}_i(\cdot, x^*, \alpha^*, \beta^*)$ *is* $(\eta, \bar{\rho}_i)$*-quasiinvex at* x^**;*

(ii) $\mathcal{F}(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho})$*-pseudoinvex at* x^**;*

(iii) $\rho^\circ + \tilde{\rho} \geqq 0$.

Then x^* *is an efficient solution of* (P)*.*

Proof. (a): Suppose to the contrary that x^* is not an efficient solution of (P). As shown in the proof of Theorem 3.4, this supposition leads to the inequalities (3.11) and (3.12) for some $\bar{x} \in \mathbb{F}$. In view of (i), this implies that for each $i \in \underline{p}$,

$$\langle D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}], \eta(\bar{x}, x^*)\rangle \leqq -\bar{\rho}_i \|\bar{x} - x^*\|^2.$$

Since $I_+ \neq \emptyset$ and for $i \in \underline{p} \backslash I_+$, $u_i^* = 0$ and $\sum_{i \in I_+} u_i^* = 1$, the above inequalities yield

$$\left\langle \sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}, \eta(\bar{x}, x^*) \right\rangle \leqq -\sum_{i \in I_+} u_i^* \bar{\rho}_i \|\bar{x} - x^*\|^2. \tag{3.14}$$

As shown in the proof of Theorem 3.1, our assumptions in (ii) and (iii) lead to (3.6) and (3.7), respectively, which when combined with (3.1) and (iv) yield

$$\left\langle \sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}, \eta(\bar{x}, x^*) \right\rangle$$
$$> -\sum_{i \in I_+} u_i^* \bar{\rho}_i \|\bar{x} - x^*\|^2.$$

which contradicts (3.14). Hence x^* is an efficient solution of (P).

(b)–(l): The proofs are similar to that of part (a). □

4. Generalized Sufficient Efficiency Criteria

In this section, we discuss several families of sufficient efficiency results under various generalized (η, ρ)-invexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain type of partitioning scheme which was originally proposed in [13] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0, J_1, \dots, J_m\}$ and $\{K_0, K_1, \dots, K_m\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$, respectively; thus, $J_\mu \subset \underline{q}$ for each $\mu \in \underline{m} \cup \{0\}$, $J_\mu \cap J_\nu = \emptyset$ for each $\mu, \nu \in \underline{m} \cup \{0\}$ with $\mu \neq \nu$, and $\cup_{\mu=0}^{m} J_\mu = \underline{q}$. Obviously, similar properties hold for $\{K_0, K_1, \dots, K_m\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $m = \max\{m_1, m_2\}$ and $J_\mu = \emptyset$ or $K_\mu = \emptyset$ for $\mu > \min\{m_1, m_2\}$

In addition, we use the real-valued functions $\Phi_i(\cdot, y, v, w, \alpha, \beta, \gamma)$, $\Phi(\cdot, y, u, v, w, \alpha, \beta, \gamma)$, and $\Lambda_t(\cdot, v, w, \gamma)$ defined, for fixed $y, u, v, w, \alpha, \beta$, and γ, on X as follows:

$$\Phi_i(x, y, v, w, \alpha, \beta, \gamma) = D_i(y)[f_i(x) + \langle \alpha^i, A_i x \rangle] - N_i(y)[g_i(x) - \langle \beta^i, B_i x \rangle] + \sum_{j \in J_0} v_j [G_j(x) + \langle \gamma^j, C_j x \rangle] + \sum_{k \in K_0} w_k H_k(x), \quad i \in \underline{p},$$

$$\Phi(x, y, u, v, w, \alpha, \beta, \gamma) = \sum_{i=1}^{p} u_i \{D_i(y)[f_i(x) + \langle \alpha^i, A_i x \rangle] - N_i(y)[g_i(x) - \langle \beta^i, B_i x \rangle]\} + \sum_{j \in J_0} v_j [G_j(x) + \langle \gamma^j, C_j x \rangle] + \sum_{k \in K_0} w_k H_k(x)$$

$$\Lambda_t(x, v, w, \gamma) = \sum_{j \in J_t} v_j [G_j(x) + \langle \gamma^j, C_j x \rangle] + \sum_{k \in K_t} w_k H_k(x), \quad t \in \underline{m}.$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized sufficiency results for (P) as follows.

THEOREM 4.1. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, and that there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1) – (3.5) hold. Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:*

(a) *(i)* $\Phi(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho}_t)$*-pseudoinvex at* x^**;*

(iii) $\bar{\rho} + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$*;*

(b) *(i)* $\Phi(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\eta, \bar{\rho})$*-pseudoinvex at* x^**;*

(ii) for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho}_t)$*-quasiinvex at* x^**;*

(iii) $\bar{\rho} + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$*;*

(c) *(i)* $\Phi(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho}_t)$*-quasiinvex at* x^**;*

(iii) $\bar{\rho} + \sum_{t=1}^m \tilde{\rho}_t > 0$*;*

(d) *(i)* $\Phi(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* $(\eta, \bar{\rho})$*-quasiinvex at* x^**;*

(ii) for each $t \in \underline{m_1}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho}_t)$*-quasiinvex at* x^**, and for each* $t \in \underline{m_2} \neq \emptyset$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho}_t)$*-pseudoinvex at* x^**, where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$*;*

(iii) $\bar{\rho} + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$.

Then x^* is an efficient solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).

(a): It is clear that (3.1) can be expressed as follows:

$$\sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\}$$
$$+ \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*)$$
$$+ \sum_{t=1}^{m} \Big\{ \sum_{j \in J_t} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_t} w_k^* \nabla H_k(x^*) \Big\} = 0. \tag{4.1}$$

Since for each $t \in \underline{m}$,

$$\begin{aligned}
&\Lambda_t(x, v^*, w^*, \gamma^*)\\
&\quad = \sum_{j\in J_t} v_j^*[G_j(x) + \langle \gamma^*, C_j x\rangle] + \sum_{k\in K_t} w_k^* H_k(x)\\
&\quad \leqq \sum_{j\in J_t} v_j^*[G_j(x) + \|\gamma^*\|_{c(j)}^* \|C_j x\|_{c(j)}] + \sum_{k\in K_t} w_k^* H_k(x)\\
&\qquad \text{(by Lemma 3.1 and nonnegativity of } v^*)\\
&\quad \leqq \sum_{j\in J_t} v_j^*[G_j(x) + \|C_j x\|_{c(j)}] + \sum_{k\in K_t} w_k^* H_k(x)\\
&\qquad \text{(by (3.4) and nonnegativity of } v^*)\\
&\quad \leqq 0 \quad \text{(by the feasibility of } x \text{ and nonnegativity of } v^*)\\
&\quad = \sum_{j\in J_t} v_j^*[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle] + \sum_{k\in K_t} w_k^* H_k(x^*)\\
&\qquad \text{(by (3.2) and feasibility of } x^*)\\
&\quad = \Lambda_t(x^*, v^*, w^*, \gamma^*),
\end{aligned}$$

it follows from (ii) that

$$\Big\langle \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*), \eta(x, x^*)\Big\rangle < -\tilde{\rho}_t \|x - x^*\|^2.$$

Summing over t, we obtain

$$\begin{aligned}
&\Big\langle \sum_{t=1}^{m} \Big\{ \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*)\Big\}, \eta(x, x^*)\Big\rangle\\
&\quad < -\sum_{t=1}^{m} \tilde{\rho}_t \|x - x^*\|^2. \qquad (4.2)
\end{aligned}$$

Combining (4.1) and (4.2), and using (iii) we get

$$\begin{aligned}
&\Big\langle \sum_{i=1}^{p} u_i^*\{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*iT}]\}\\
&\quad + \sum_{j\in J_0} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*jT}]\\
&\quad + \sum_{k\in K_0} w_k^* \nabla H_k(x^*), \eta(x, x^*)\Big\rangle > \sum_{t=1}^{m} \tilde{\rho}_t \|x - x^*\|^2 \geqq -\bar{\rho}\|x - x^*\|^2, \qquad (4.3)
\end{aligned}$$

which by virtue of (i) implies that $\Phi(x, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) \geqq \Phi(x^*, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) = 0$, where the equality follows from the definitions of $N_i(x^*)$ and $D_i(x^*)$, $i \in \underline{p}$, (3.2), (3.5), and feasibility of x^*.

Therefore, bearing in mind that $u^* > 0$, $N_i(x^*) \geqq 0$, $D_i(x^*) > 0$, $i \in \underline{p}$, and $v^* \geqq 0$, we have

$$\begin{aligned}
0 &\leqq \Phi(x, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) \\
&\leqq \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(x) + \|\alpha^{*i}\|_{a(i)}^* \|A_i x\|_{a(i)}] \\
&\quad - N_i(x^*)[g_i(x) - \|\beta^{*i}\|_{b(i)}^* \|B_i x\|_{b(i)}]\} \\
&\quad + \sum_{j \in J_0} v_j^* [G_j(x) + \|\gamma^{*j}\|_{c(j)}^* \|C_j x\|_{c(j)}] \\
&\quad \text{(by Lemma 3.1 and feasibility of } x) \\
&\leqq \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(x) + \|A_i x\|_{a(i)}] - N_i(x^*)[g_i(x) - \|B_i x\|_{b(i)}]\} \\
&\quad + \sum_{j \in J_0} v_j^* [G_j(x) + \|C_j x\|_{c(j)}] \quad \text{(by (3.3) and (3.4))} \\
&\leqq \sum_{i=1}^{p} u_i^* \{D_i(x^*)[f_i(x) + \|A_i x\|_{a(i)}] - N_i(x^*)[g_i(x) - \|B_i x\|_{b(i)}]\} \\
&\quad \text{(by the feasibility of } x),
\end{aligned}$$

which is precisely (3.8). In the proof of part (a) of Theorem 3.1 it was shown that this inequality leads to the conclusion that x^* is an efficient solution of (P).

(b): Proceeding as in the proof of part (a), we see that (ii) leads to the following inequality:

$$\begin{aligned}
&\left\langle \sum_{t=1}^{m} \left\{ \sum_{j \in J_t} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_t} w_k^* \nabla H_k(x^*) \right\}, \eta(x, x^*) \right\rangle \\
&\quad \leqq - \sum_{t=1}^{m} \tilde{\rho}_t \|x - x^*\|^2.
\end{aligned}$$

Now combining this inequality with (4.1) and using (iii), we obtain

$$\begin{aligned}
&\Big\langle \sum_{i=1}^{p} u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*iT}]\} \\
&\quad + \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*jT}] \\
&\quad + \sum_{k \in K_0} w_k^* \nabla H_k(x^*), \eta(x, x^*) \Big\rangle \geqq \sum_{t=1}^{m} \tilde{\rho}_t \|x - x^*\|^2 \geqq -\bar{\rho} \|x - x^*\|^2,
\end{aligned}$$

which by virtue of (i) implies that $\Phi(x, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) \geqq \Phi(x^*, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$. The rest of the proof is identical to that of part (a).

(c) and (d): The proofs are similar to those of parts (a) and (b). □

THEOREM 4.2. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, and that there exist $u^* \in U_0$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1)–(3.5) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at x^*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(b) (i) *for each $i \in I_+$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is strictly $(\eta, \tilde{\rho}_t)$-pseudoinvex at x^*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(c) (i) *for each $i \in I_+$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at x^*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t > 0$;

(d) (i) *for each $i \in I_{1+}$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*, and for each $i \in I_{2+}$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at x^*, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is strictly $(\eta, \tilde{\rho}_t)$-pseudoinvex at x^*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(e) (i) *for each $i \in I_{1+} \neq \emptyset$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is strictly $(\eta, \bar{\rho}_i)$-pseudoinvex at x^*, and for each $i \in I_{2+}$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at x^*, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is $(\eta, \tilde{\rho}_t)$-quasiinvex at x^*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(f) (i) *for each $i \in I_+$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\eta, \bar{\rho}_i)$-quasiinvex at x^*;*

(ii) *for each* $t\in \underline{m}_1\neq\emptyset$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho}_t)$*-pseudoinvex at* x^**, and for each* $t\in \underline{m}_2$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho}_t)$*-quasiinvex at* x^**, where* $\{\underline{m}_1, \underline{m}_2\}$ *is a partition of* $\underline{m}$*;*

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$*;*

(g) *(i)* *for each* $i\in I_{1+}$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* $(\eta, \bar{\rho}_i)$*-pseudoinvex at* x^**, and for each* $i\in I_{2+}$, $\Phi_i(\cdot, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* $(\eta, \bar{\rho}_i)$*-quasiinvex at* x^**, where* $\{I_{1+}, I_{2+}\}$ *is a partition of* I_+*;*

(ii) *for each* $t\in \underline{m}_1$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\eta, \tilde{\rho}_t)$*-pseudoinvex at* x^**, and for each* $t\in \underline{m}_2$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\eta, \tilde{\rho}_t)$*-quasiinvex at* x^**, where* $\{\underline{m}_1, \underline{m}_2\}$ *is a partition of* $\underline{m}$*;*

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$*;*

(iv) $I_{1+}\neq\emptyset$, $\underline{m_1}\neq\emptyset$, *or* $\sum_{i\in I_+} u_i^*\bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t > 0$.

Then x^* *is an efficient solution of* (P).

Proof. (a): Suppose to the contrary that x^* is not an efficient solution of (P). As shown in the proof of Theorem 3.4, this supposition leads to (3.9) and (3.10) for some $\bar{x}\in\mathbb{F}$. Keeping in mind that $N_i(x^*)\geqq 0$, $D_i(x^*)>0$, $i\in \underline{p}$, and $v^*\geqq 0$, we see that for each $i\in I_+$,

$$
\begin{aligned}
&\Phi_i(\bar{x}, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)\\
&\leqq D_i(x^*)[f_i(\bar{x}) + \|\alpha^{*i}\|^*_{a(i)}\|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|\beta^{*i}\|^*_{b(i)}\|B_i\bar{x}\|_{b(i)}]\\
&\quad + \sum_{j\in J_0} v_j^*[G_j(\bar{x}) + \|\gamma^{*j}\|^*_{c(j)}\|C_j\bar{x}\|_{c(j)}]\\
&\quad \text{(by Lemma 3.1 and feasibility of } \bar{x})\\
&\leqq D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}]\\
&\quad + \sum_{j\in J_0} v_j^*[G_j(\bar{x}) + \|C_j\bar{x}\|_{c(j)}] \quad \text{(by (3.3) and (3.4))}\\
&\leqq D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}]\\
&\quad \text{(by the feasibility of } \bar{x})\\
&\leqq 0 \text{ (by (3.9) and (3.10))}\\
&= D_i(x^*)[f_i(x^*) + \langle\alpha^{*i}, A_i x^*\rangle] - N_i(x^*)[g_i(x^*) - \langle\beta^{*i}, B_i x^*\rangle]\\
&\quad + \sum_{j\in J_0} v_j^*[G_j(x^*) + \langle\gamma^{*j}, C_j x^*\rangle]\\
&\quad + \sum_{k\in K_0} w_k^* H_k(x^*) \quad \text{(by the definitions of } N_i(x^*) \text{ and } D_i(x^*),
\end{aligned}
$$

$$\begin{aligned} & i \in \underline{p}, (3.2), (3.5), \text{ and feasibility of } x^*) \\ & = \Phi_i(x^*, x^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*), \end{aligned}$$

which in view of (i) implies that for each $i \in I_+$,

$$\begin{aligned} &\Big\langle D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}] + \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) \\ &+ C_j^T \gamma^{*j}] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*), \eta(\bar{x}, x^*)\Big\rangle < -\bar{\rho}_i \|\bar{x} - x^*\|^2. \end{aligned}$$

Since $u^* \geqq 0$, $u_i^* = 0$ for each $i \in \underline{p} \backslash I_+$, and $\sum_{i=1}^p u_i^* = 1$, the above inequalities yield

$$\begin{aligned} &\Big\langle \sum_{i=1}^p u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\} \\ &+ \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] \\ &+ \sum_{k \in K_0} w_k^* \nabla H_k(x^*), \eta(\bar{x}, x^*)\Big\rangle < -\sum_{i \in I_+} u_i^* \bar{\rho}_i \|\bar{x} - x^*\|^2. \end{aligned} \tag{4.4}$$

As seen in the proof of Theorem 4.1, our assumptions in (ii) lead to

$$\begin{aligned} &\Big\langle \sum_{t=1}^m \Big\{ \sum_{j \in J_t} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_t} w_k^* \nabla H_k(x^*) \Big\}, \eta(\bar{x}, x^*) \Big\rangle \\ &\leqq -\sum_{t=1}^m \tilde{\rho}_t \|\bar{x} - x^*\|^2, \end{aligned}$$

which when combined with (4.1), results into

$$\begin{aligned} &\Big\langle \sum_{i=1}^p u_i^* \{D_i(x^*)[\nabla f_i(x^*) + A_i^T \alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T \beta^{*i}]\} \\ &+ \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] \\ &+ \sum_{k \in K_0} w_k^* \nabla H_k(x^*), \eta(\bar{x}, x^*)\Big\rangle \geqq \sum_{t=1}^m \tilde{\rho}_t \|\bar{x} - x^*\|^2. \end{aligned}$$

In view of (iii), this inequality contradicts (4.4). Hence, x^* is an efficient solution of (P).

(b)–(g): The proofs are similar to that of part (a). □

In the remainder of this section we present another collection of sufficiency results which are somewhat different from those stated in Theorems 4.1 and 4.2. These results are formulated by utilizing a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$, and by placing appropriate generalized (η, ρ)-invexity requirements on certain combinations of the functions $\mathcal{E}_i(\cdot, x^*, \alpha, \beta)$, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$. The particular partitioning method used here was originally utilized by Yang [19] for constructing a general dual problem for a multiobjective fractional programming problem.

Let $\{I_0, I_1, \ldots, I_\ell\}$ be a partition of $\underline{p}$ such that $L = \{0, 1, 2, \ldots, \ell\} \subset M = \{0, 1, \ldots, m\}$, and let the function $\Pi_t(\cdot, y, u, v, w, \alpha, \beta, \gamma) : X \to \mathbb{R}$ be defined, for fixed $y, u, v, w, \alpha, \beta$, and γ, by

$$\Pi_t(x, y, u, v, w, \alpha, \beta, \gamma) = \sum_{i \in I_t} u_i \{D_i(y)[f_i(x) + \langle \alpha^i, A_i x\rangle] \\ - N_i(y)[g_i(x) - \langle \beta^i, B_i x\rangle]\} \\ + \sum_{j \in J_t} v_j [G_j(x) + \langle \gamma^j, C_j x\rangle] + \sum_{k \in K_t} w_k H_k(x), \ t \in \underline{m}.$$

THEOREM 4.3. *Let $x^* \in \mathbb{F}$ and assume that $N_i(x^*) \geqq 0$, $i \in \underline{p}$, and that there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (3.1)–(3.5) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) for each $t \in L$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is strictly (η, ρ_t)-pseudoinvex at x^*;
(ii) for each $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is (η, ρ_t)-quasiinvex at x^*;
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(b) (i) for each $t \in L$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly (η, ρ_t)-quasiinvex at x^*;
(ii) for each $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is strictly (η, ρ_t)-pseudoinvex at x^*;
(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(c) (i) for each $t \in L$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly (η, ρ_t)-quasiinvex at x^*;
(ii) for each $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is (η, ρ_t)-quasiinvex at x^*;
(iii) $\sum_{t \in M} \rho_t > 0$;

(d) (i) for each $t \in L_1$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is strictly (η, ρ_t)-pseudoinvex at x^*, and for each $t \in L_2$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly (η, ρ_t)-quasiinvex at x^*, where $\{L_1, L_2\}$ is a partition of L;

(ii) for each $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* (η, ρ_t)*-pseudoinvex at* x^**;*

(iii) $\sum_{t \in M} \rho_t \geqq 0$*;*

(e) (i) for each $t \in L_1 \neq \emptyset$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* (η, ρ_t)*-pseudoinvex at* x^**, and for each* $t \in L_2$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* (η, ρ_t)*-quasiinvex at* x^**, where* $\{L_1, L_2\}$ *is a partition of* L*;*

(ii) for each $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* (η, ρ_t)*-quasiinvex at* x^**;*

(iii) $\sum_{t \in M} \rho_t \geqq 0$*;*

(f) (i) for each $t \in L$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* (η, ρ_t)*-quasiinvex at* x^**;*

(ii) for each $t \in (M \setminus L)_1 \neq \emptyset$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* (η, ρ_t)*-pseudoinvex at* x^**, and for each* $t \in (M \setminus L)_2$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* (η, ρ_t)*-quasiinvex at* x^**, where* $\{(M \setminus L)_1, (M \setminus L)_2\}$ *is a partition of* $M \setminus L$*;*

(iii) $\sum_{t \in M} \rho_t \geqq 0$*;*

(g) (i) for each $t \in L_1$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* (η, ρ_t)*-pseudoinvex at* x^**, and for each* $t \in L_2$, $\Pi_t(\cdot, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* (η, ρ_t)*-quasiinvex at* x^**, where* $\{L_1, L_2\}$ *is a partition of* L*;*

(ii) for each $t \in (M \setminus L)_1$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* (η, ρ_t)*-pseudoinvex at* x^**, and for each* $t \in (M \setminus L)_2$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* (η, ρ_t)*-quasiinvex at* x^**, where* $\{(M \setminus L)_1, (M \setminus L)_2\}$ *is a partition of* $M \setminus L$*;*

(iii) $\sum_{t \in M} \rho_t \geqq 0$*;*

(iv) $L_1 \neq \emptyset$, $(M \setminus L)_1 \neq \emptyset$, *or* $\sum_{t \in M} \rho_t > 0$.

Then x^* *is an efficient solution of* (P)*.*

Proof. (a): Suppose to the contrary that x^* is not an efficient solution of (P). As seen in the proof of Theorem 3.4, this supposition leads to the inequalities

$$D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}] \leqq 0, \quad i \in \underline{p},$$

with strict inequality holding for at least one index $m \in \underline{p}$, for some $\bar{x} \in \mathbb{F}$. Therefore, for each $t \in L$,

$$\sum_{i \in I_t} u_i^* \{D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}]\} \leqq 0, \qquad (4.5)$$

Now using this inequality, we see that

$$\begin{aligned}
&\Pi_t(\bar{x}, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) \\
&\leqq \sum_{i\in I_t} u_i^*\{D_i(x^*)[f_i(\bar{x}) + \|\alpha^{*i}\|_{a(i)}^* \|A_i\bar{x}\|_{a(i)}] \\
&\quad - N_i(x^*)[g_i(\bar{x}) - \|\beta^{*i}\|_{b(i)}^* \|B_i\bar{x}\|_{b(i)}]\} + \sum_{j\in J_t} v_j^*[G_j(\bar{x}) + \|\gamma^{*j}\|_{c(j)}^* \|C_j x\|_{c(j)}] \\
&\quad \text{(by Lemma 3.1 and feasibility of } \bar{x}) \\
&\leqq \sum_{i\in I_t} u_i^*\{D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}]\} \\
&\quad + \sum_{j\in J_t} v_j^*[G_j(\bar{x}) + \|C_j\bar{x}\|_{c(j)}] \quad \text{(by (3.3) and (3.4))} \\
&\leqq \sum_{i\in I_t} u_i^*\{D_i(x^*)[f_i(\bar{x}) + \|A_i\bar{x}\|_{a(i)}] - N_i(x^*)[g_i(\bar{x}) - \|B_i\bar{x}\|_{b(i)}]\} \\
&\quad \text{(by the feasibility of } \bar{x}) \\
&\leqq 0 \quad \text{(by (4.5))} \\
&= \sum_{i\in I_t} u_i^*\{D_i(x^*)[f_i(x^*) + \langle \alpha^{*i}, A_i x^*\rangle] - N_i(x^*)[g_i(x^*) - \langle \beta^{*i}, B_i x^*\rangle]\} \\
&\quad + \sum_{j\in J_t} v_j^*[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle] \\
&\quad + \sum_{k\in K_t} w_k^* H_k(x^*) \quad \text{(by the definitions of } D_i(x^*) \text{ and } N_i(x^*), \\
&\quad i \in \underline{p}, \text{(3.2), (3.5), and feasibility of } x^*) \\
&= \Pi_t(x^*, x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*),
\end{aligned}$$

which in view of (i) implies that

$$\begin{aligned}
&\Big\langle \sum_{i\in I_t} u_i^*\{D_i(x^*)[\nabla f_i(x^*) + A_i^T\alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T\beta^{*i}]\} \\
&\quad + \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T\gamma^{*j}] \\
&\quad + \sum_{k\in K_t} w_k^*\nabla H_k(x^*), \eta(\bar{x}, x^*)\Big\rangle < -\rho_t\|\bar{x} - x^*\|^2.
\end{aligned}$$

Adding the above inequalities, we get

$$\Big\langle \sum_{i=1}^{p} u_i^*\{D_i(x^*)[\nabla f_i(x^*) + A_i^T\alpha^{*i}] - N_i(x^*)[\nabla g_i(x^*) - B_i^T\beta^{*i}]\}$$

$$+\sum_{t\in L}\Big\{\sum_{j\in J_t} v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]$$
$$+\sum_{k\in K_t} w_k^*\nabla H_k(x^*)\Big\},\eta(\bar{x},x^*)\Big\rangle < -\sum_{t\in L}\rho_t\|\bar{x}-x^*\|^2. \tag{4.6}$$

As shown in the proof of Theorem 4.1, for each $t\in M\backslash L$, $\Lambda_t(\bar{x},v^*,w^*,\gamma^*)\leqq \Lambda_t(x^*,v^*,w^*,\gamma^*)$, which in view of (ii) implies that

$$\Big\langle\sum_{j\in J_t} v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k\in K_t} w_k^*\nabla H_k(x^*),\eta(\bar{x},x^*)\Big\rangle\leqq -\rho_t\|\bar{x}-x^*\|^2.$$

Summing over t, we obtain

$$\Big\langle\sum_{t\in M\backslash L}\Big\{\sum_{j\in J_t} v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k\in K_t} w_k^*\nabla H_k(x^*)\Big\},\eta(\bar{x},x^*)\Big\rangle$$
$$\leqq -\sum_{t\in M\backslash L}\rho_t\|\bar{x}-x^*\|^2. \tag{4.7}$$

Now combining (4.6) and (4.7) and using (iii), we obtain

$$\Big\langle\sum_{i=1}^{p} u_i^*\{D_i(x^*)[\nabla f_i(x^*)+A_i^T\alpha^{*i}]-N_i(x^*)[\nabla g_i(x^*)-B_i^T\beta^{*i}]\}$$
$$+\sum_{j=1}^{q} v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]$$
$$+\sum_{k=1}^{r} w_k^*\nabla H_k(x^*),\eta(\bar{x},x^*)\Big\rangle < -\sum_{t=1}^{m}\rho_t\|\bar{x}-x^*\|^2\leqq 0,$$

which contradicts (3.1). Therefore, x^* is an efficient solution of (P).

(b)–(g): The proofs are similar to that of part (a). □

Each one of the 18 sets of conditions given in Theorems 4.1–4.3 can be viewed as a family of sufficient efficiency conditions whose members can easily be identified by appropriate choices of the partitioning sets J_μ, K_μ, $\mu\in\underline{m}\cup\{0\}$, and I_ν, $\nu\in\underline{\ell}\cup\{0\}$. These sufficiency conditions along with their numerous special cases and variants provide a multitude of global optimality and efficiency criteria for several classes of single- and multiple-objective nonlinear programming problems with and without arbitrary norms and square roots of positive semidefinite quadratic forms.

5. Concluding Remarks

In this paper, we have established, in a unified framework, a fairly large number of sets of global semiparametric sufficient efficiency conditions under a variety of generalized (η, ρ)-invexity assumptions for a multiobjective fractional programming problem containing arbitrary norms (and square roots of positive semidefinite quadratic forms). Each one of these sufficiency results can easily be modified and restated for each one of the ten special cases of the prototype problem (P) designated as $(P1)$–$(P10)$ in Section 1, and hence they collectively subsume a truly vast number of sufficient optimality and efficiency results previously established by different methods for various classes of nonlinear programming problems with multiple, fractional, and conventional objective functions. Furthermore, the style and techniques employed in this paper can be utilized for developing similar results for some other classes of optimization problems involving more general types of convex functions. These include discrete and continuous minmax fractional programming problems, various classes of semiinfinite programming problems, and certain types of continuous-time programming problems.

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